A Quasilinear-Approximation Model for Turbulent Shear Flow

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Within the small turbulence Reynolds number approximation, closed form solutions are found for the equations governing the random fluctuations of hydrodynamic turbulence, under the assumption of special time-dependent uniform or sheared mean flow profiles. Constant, transient, and oscillatory flows are considered. It is found that in most cases the velocity correlation and, in turn, the turbulent kinetic energy are attenuated exponentially with respect to time. For sheared flows it is possible that the turbulent energy grows with time for a while but eventually decays again.

I. Introduction

THE traditional way of approaching turbulence problems in engineering applications is via some closure assumption on the second- or third-order correlation.[1,2,3] Arbitrary constants and functions are then determined by comparison to experiments. A very large number of papers on the topic have appeared in the last two decades, suggesting further improvements and generalizing the theories for unconventional flows. Detailed accounts and brief descriptions can be found in recent review articles.[4-7] On the other hand the approach of studying the problem through a mathematical approximation has not been pursued very actively. True, such methods are based on the small disturbance hypothesis, which may not be uniformly valid, especially in boundary layers where the bursting phenomena are clearly violating this assumption. Nevertheless some interesting features of the flow can be deduced and, with the advent of the modern computing machines, it is possible that such methods may prove more meaningful, useful, and more widely applicable.

Statistical studies of decaying turbulence were quite successful in predicting turbulent dissipation and transfer of energy between eddies of various sizes for isotropic flows.[8,9] More recently Deissler[10] and Hasen[11] have extended such theories to flows with linear mean profiles. Assuming homogeneous flow, that is assuming that the second-order correlation, $\langle u_i(x,t)u_j(x,t)\rangle$, where u_j is the velocity fluctuation vector, depends only on the vector distance between two points, $x-x'$ and dropping the triple correlations, they solve the differential equation for $\langle u_i u_j\rangle$. The present authors[12] have recently developed a statistical approach along the lines of the Weinstock-Balescu-Misguich formalism.[13-15] In the weak-coupling limit one arrives at a complete set of closed equations for $\langle u_j\rangle$ and $\langle u_i u_j'\rangle$, corresponding to Kraichnan's direct-interaction approximation.[16,17]

In the present paper a new method for examining linear and nonlinear stochastic differential equations is employed in order to sudy the possible interaction between a uniform or sheared mean flow and turbulence. In particular, time dependent mean flows are considered that involve transient or oscillatory deterministic disturbances of the mean flow. Such flows are often encountered in engineering boundary layers over helicopter blades, or turbomachinery cascades, etc. In the next section we present some basic concepts of the theory. Section III is concerned with uniform but unsteady mean flow. Section IV deals with steady shear flow and consists of a generalization of the work of Deissler[10] and Hasen[11] to nonhomogeneous flow. Finally in Sec. V we study the coupling of the random fluctuations with unsteady sheared mean flow.

II. The Governing Equations

The incompressibility condition and the Navier-Stokes equations in tensor notation take the form

$$\partial \tilde{u}_i / \partial x_i = 0 \tag{1}$$

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_i} + \nu \nabla^2 \tilde{u}_i \tag{2}$$

where $\tilde{u}_i$ and $\tilde{p}$ are the velocity vector and the pressure divided by density respectively, x_i is the position vector and ν is the kinematic viscosity.

Vector and tensor quantities will be denoted with one or two indices respectively. The indicial notation and the familiar summation convention will be used throughout for all dependent variables. Position vectors are denoted as boldface when they appear as independent variables within a parenthesis and with proper indices when they are used to denote differentiation. Taking the divergence of Eq. (2) and using Eq. (1) one may arrive at an equation for the pressure[17]

$$\nabla^2 \tilde{p} = -\frac{\partial^2}{\partial x_i \partial x_j} (\tilde{u}_i \tilde{u}_j) \tag{3}$$

which, in turn, can be used to eliminate $\tilde{p}$ from the Navier-Stokes equations. Let us assume now that we can decompose the velocity and pressure fields as follows:

$$\tilde{u}_i = U_i + u_i \tag{4}$$

$$\tilde{p} = P + p \tag{5}$$

where U_i and P are the ensemble average of the velocity and pressure, respectively, viz.

$$U_i = \langle \tilde{u}_i \rangle, \quad P = \langle \tilde{p} \rangle \tag{6}$$

and u_i, p represents the random fluctuations of the velocity and the pressure respectively. We now assume that the turbulent fluctuations are small, that is, we propose to consider here only flows with low turbulence Reynolds number.[8,9] This assumption is not too unrealistic even though it is well known by now that some turbulent fields involve quite large random fluctuations.

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Substituting Eqs. (4) and (5) into Eqs. (2) and (3) and taking an ensemble average, yields the familiar equations that govern the mean flow. These equations are coupled with the fluctuations via the Reynolds stress. The equations that govern the random fluctuations read

$$\frac{\partial u_i}{\partial t} = (-U_j\frac{\partial}{\partial x_j} + \nu\nabla^2)u_i + u_j\frac{\partial U_i}{\partial x_j} - \frac{\partial p}{\partial x_i} \tag{7}$$

$$\nabla^2 p = -\frac{\partial}{\partial x_i}(U_j\frac{\partial u_i}{\partial x_j} + u_j\frac{\partial U_i}{\partial x_j}) = -\frac{2\partial^2}{\partial x_i \partial x_j}(U_j u_i) \tag{8}$$

In these equations quantities nonlinear in terms of the small fluctuations have been dropped. This approach therefore, is equivalent to Deissler's[10] who works with the equations for the velocity and pressure correlation but disregards triple correlations.

Equation (8) can be solved in general using an appropriate Green's function. For an infinite domain, the well-known solution to Poisson's equation is given by

$$p(x,t) = -\int_{-\infty}^{\infty} \frac{2}{|x-x'|}\frac{\partial^2(U_j u_i)}{\partial x_i' \partial x_j'}\mathrm{d}x' \tag{9}$$

Solutions to Eqs. (7) now can be constructed in general as follows (cf. Appendix A)

$$u_i(x,t) = G(x,t_0)u_i(x,t_0) + \int_{t_0}^{t} G(x,t,s)u_j(x,s)\frac{\partial U_i}{\partial x_j}\mathrm{d}s + \int_{t_0}^{t} G(x,t,s)\frac{\partial p(x,s)}{\partial x_i}\mathrm{d}s \tag{10}$$

where G is an operator commonly referred to in literature as propagator. The above expressions are integral equations for the unknown $u_i(x_i,t)$. In the sections that follow we shall consider several special cases, whereby the term $u_j\partial U_i/\partial x_j$ vanishes and the first convolution integral of Eq. (10) is eliminated. For such cases, calculations of a second moment can be performed if the initial distribution of the velocity and its second moment is known

$$\begin{aligned}\Gamma_{ij}(x,x',t) &= \langle u_i(x,t)u_j(x',t)\rangle \\ &= G(x,t,t_0)G(x',t,t_0)\Gamma_{ij}(x,x',t_0) \\ &+ \int_{t_0}^{t} G(x,t,s)G(x',t,t_0)\langle\frac{\partial p(x,s)}{\partial x_i}u_j(x',t_0)\rangle\mathrm{d}s \\ &+ \int_{t_0}^{t} G(x',t,s)G(x,t,t_0)\langle u_i(x,t_0)\frac{\partial p(x',s)}{\partial x_j}\rangle\mathrm{d}s \\ &+ \int_{t_0}^{t}\int_{t_0}^{t} G(x,t,s)G(x',t,s')\langle\frac{\partial p(x,s)}{\partial x_i}\frac{\partial p(x',s')}{\partial x_j}\rangle\mathrm{d}s\mathrm{d}s'\end{aligned} \tag{11}$$

III. Uniform Unsteady Mean Flow

Let us assume that the ensemble average of the velocity field is given by

$$\langle\tilde{u}_i\rangle = U_i^\infty f(t) \tag{12}$$

where U_i^∞ is a constant vector and $f(t)$ an arbitrary function of time. Equation (7) then reduces to

$$\frac{\partial u_i}{\partial t} = (-U_j^\infty f(t)\frac{\partial}{\partial x_j} + \nu\nabla^2)u_i - \frac{\partial p}{\partial x_i} \tag{13}$$

It should be noted that for a constant U_i, the right-hand side of Eq. (8) vanishes by virtue of the continuity equation, and, therefore, p satisfies Laplace's equation. For the infinite domain we are considering and assuming that pressure fluctuations are finite at infinity, the only nonsingular solution for p is zero or an arbitrary function of time only, throughout the space; that is, $p=p(t)$. As a result, the pressure disappears from Eqs. (7). The general solution to these equations for an infinite domain, without the pressure term on the right-hand side, is given by

$$u_i(x,t) = G(x,t)u_i(x,0) \tag{14}$$

In the sequel we will consider $t_0=0$ and for brevity we will omit t_0 from the arguments of the propagator G, i.e, $G(x,t,0) \equiv G(x,t)$. The operator $G(x,t)$ is given in general in terms of an infinite series

$$G(x,t) = \exp\left[\sum_{n=1}^{\infty}\Delta_n(t)\right] \tag{15}$$

(cf. Appendix A); however, in the particular case we are considering, all Δ_n for $n\leq 2$ vanish since the operator $-U_j^\infty f(t)\partial/\partial x_j + \nu\nabla^2$ commutes with itself. Thus,

$$\Delta_1(t) = \int_0^t(-U_j^\infty f(t)\partial/\partial x_j + \nu\nabla^2)\mathrm{d}t = -U_j^\infty g(t)\frac{\partial}{\partial x_j} + \nu t\nabla^2 \tag{16}$$

where

$$g(t) = \int_0^t f(t)\mathrm{d}t$$

It should be noted that since the operators $\partial/\partial x_j$ and ∇^2 commute, the exponential term in Eq. (15) can be factored (cf. Appendix B) and the solution to our problem reads

$$u_i(x,t) = e^{-U_j^\infty g(t)\partial/\partial x_j}e^{\nu t\nabla^2}u(x,0) \tag{17}$$

Equation (17) represents a closed form solution for the instantaneous velocity fluctuation. To bring this result into a more familiar form, let us calculate the velocity correlation

$$\Gamma_{ij}(x,x',t) = \langle u_i(x,t)u_j(x',t)\rangle \tag{18}$$

Since the operator G is deterministic, the ensemble average simply gives

$$\begin{aligned}\Gamma_{ij}(x,x',t) &= G(x,t)G(x',t)\langle u_i(x,0)u_j(x',0)\rangle \\ &= \exp\left[-U_i^\infty g(t)\left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i'}\right)\right] \\ &\exp[\nu t(\nabla_x^2 + \nabla_{x'}^2)]\Gamma_{ij}(x,x',0)\end{aligned} \tag{19}$$

Several simple initial conditions will now be assumed:

Purely Random Field

$\Gamma_{ij}(x_i,x_j',0) = A_{ij}\delta(x_i - x_i')$; where A_{ij} is a constant and δ is the Dirac delta function. Equation (19), then, via Eqs. (B3) and (B4), becomes

$$\Gamma_{ij}(x,x',t) = \frac{A_{ij}H(t)}{(8\pi\nu t)^{3/2}}\exp\left[-\frac{(x-x')^2}{8\nu t}\right] \tag{20}$$

To facilitate comparison with familiar results in the theory of turbulence, we introduce next the conventional center-of-mass and relative position variables

$$R = ½(x+x'), \quad r = x - x' \tag{21}$$

In terms of these variables, the velocity correlation becomes

$$\Gamma_{ij}(r,R,t) = \frac{A_{ij}H(t)}{(8\pi\nu t)^{3/2}}e^{(-r^2/8\nu t)} \tag{22}$$

It is seen that the time dependence of the mean flow does not affect the random fluctuations. Moreover, the homogeneity property, implicit in the initial condition, is preserved for $t>0$. Thus the above solution reduces to the well-known results of Batchelor[8] and Deissler[9] for steady mean flows. One simply needs to take a Fourier transform in order to arrive at

$$\Gamma_{ij}(\boldsymbol{k},\boldsymbol{R},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\cdot}\Gamma_{ij}(\boldsymbol{r},\boldsymbol{R},t)\,\mathrm{d}\boldsymbol{r}=\frac{A_{ij}H(t)}{(4\pi)^{3/2}}e^{(-2\nu k^2 t)} \tag{23}$$

The energy spectrum now can be derived by contracting the correlation tensor and integrating over the surface of a sphere in the k-space. One can immediately see that the turbulent energy decays exponentially with time.

Gaussian Field

$\Gamma_{ij}(x,x',0)=A_{ij}\exp[-a(x+x')^2-b(x-x')^2]$; where a and b are positive constants. Note that in terms of the coordinates of Eq. (21), the energy is initially distributed in a quassian form: $\Gamma_{ji}(x,x,0)=A_{ij}\exp(-4ax^2)$. By virtue of the formulas of Appendix B one can bring Eq. (19) in the form

$$\Gamma_{ij}(\boldsymbol{r},\boldsymbol{R},t)=\frac{A_{ij}H(t)}{[(8a\nu t+1)(8b\nu t+1)]^{3/2}}$$
$$\exp\left[-\frac{1}{8\nu t}\left(1-\frac{1}{1+8\nu bt}\right)r^2-\frac{1}{2\nu t}\left(1-\frac{1}{1+8\nu at}\right)\left(\boldsymbol{R}-U^{\infty}g(t)\right)^2\right] \tag{24}$$

Consider first the special case where the mean flow does not fluctuate with time. Specifically, we let $f(t)=1$, corresponding to which $g(t)=t$. The gaussian distribution of the energy is then diffused in space, as time increases

$$\Gamma_{ij}(0,\boldsymbol{R},t)=\frac{A_{ij}H(t)}{[(8a\nu t+1)(8a\nu t+1)]^{3/2}}$$
$$\exp\left[-\frac{1}{2\nu t}\left(1-\frac{1}{1+8\nu at}\right)\left(\boldsymbol{R}-U^{\infty}t\right)^2\right] \tag{25}$$

Next consider the special case of a mean flow that fluctuates harmonically about a constant, that is $f(t)=1+\beta\sin(\mathrm{m}_0 t)$. Here β is the relative amplitude and ω_0 is the constant frequency of the externally imposed deterministic fluctuation. This type of time dependence is very common in engineering applications, for example, in flows through turbomachinery cascades, flows over helicopter blades or internal flows through blood vessels, etc. For this particular form of the dependence on time, Eq. (24) becomes

$$\Gamma_{ij}(\boldsymbol{r},\boldsymbol{R},t)=\frac{A_{ij}H(t)}{[(8a\nu t+1)(8b\nu t+1)]^{3/2}}$$
$$\exp\left\{-\frac{1}{8\nu t}\left(1-\frac{1}{1+8\nu bt}\right)r^2-\frac{1}{2\nu t}\left(1-\frac{1}{1+8\nu at}\right)\left[\boldsymbol{R}-U^{\infty}(t+\frac{\beta}{\omega_0})+U^{\infty}\frac{\beta}{\omega_0}\cos(\omega_0 t)\right]^2\right\} \tag{26}$$

This case is characterized by a fluctuation superimposed on the decaying process. Moreover, the fluctuating part has a 90° phase advance with respect to the mean flow and its amplitude decreases with the increase of the frequency of the forced oscillation. This means that high frequency oscillations would be filtered out and would never excite the turbulent field.

IV. Steady Sheared Mean Flow

Consider a mean flow with a linear velocity profile, a case extensively studied both theoretically and experimentally.[17,18] Specifically, let

$$U_1(x,y,z)=ay,\ U_2(x,y,z)=0,\ U_3(x,y,z)=0 \tag{27}$$

We decided in this section to sacrifice elegance for the sake of clarity, by using a mixed notation and replacing occasionally the coordinates x_1, x_2, x_3 and the velocities u_1, u_2, u_3 with the symbols x,y and z and u, v, w, respectively. The differential equations that govern the fluctuating velocity components, [cf. Eqs. (7)], within the limitations of our weak turbulence assumption, become

$$\frac{\partial u}{\partial t}=(-ay\frac{\partial}{\partial x}+\nu\nabla^2)u-av-\frac{\partial p}{\partial x} \tag{28}$$

$$\frac{\partial v}{\partial t}=(-ay\frac{\partial}{\partial x}+\nu\nabla^2)v-\frac{\partial p}{\partial y} \tag{29}$$

$$\frac{\partial w}{\partial t}=(-ay\frac{\partial}{\partial x}+\nu\nabla^2)w-\frac{\partial p}{\partial z} \tag{30}$$

A general solution to the above set of equations can be written in terms of convolution integrals to account for the forcing functions $\partial p/\partial x$, $\partial p/\partial y$, and $\partial p/\partial z$. However, meaningful physical interpretation, cannot be extracted without tedious numerical calculations of the integrals involved. In the present section instead, we shall seek closed form analytical solutions to simplified equations, namely:

$$\frac{\partial u}{\partial t}=(-ay\frac{\partial}{\partial x}+\nu\nabla^2)u-av \tag{31}$$

$$\frac{\partial v}{\partial t}=(-ay\frac{\partial}{\partial x}+\nu\nabla^2)v \tag{32}$$

$$\frac{\partial w}{\partial t}=(-ay\frac{\partial}{\partial x}+\nu\nabla^2)w \tag{33}$$

This is certainly not fully justified, and the new equations cannot be used to study precisely the phenomenon of hydrodynamic turbulence. Nevertheless, we believe that the mechanism of turbulent interaction between the mean sheared flow and the random fluctuations is not completely ignored and, as a result, Eqs. (31-33) constitute an approximate model for the phenomenon of weak turbulence. Moreover, Deissler[10] pointed out that the contribution of pressure for this particular problem is confined to the coefficient of exponential functions and, therefore, does not participate in the exponential growth or decay of the disturbances. In fact, Burgers et al.[19] indicated that for the present case the pressure terms disappear altogether from the contracted form of the velocity correlation equation.

The propagator for all three differential expressions in Eq. (31-33) is given by (cf. Appendix A)

$$G(x,t)=\exp(\Delta_t)=\exp\int_0^t(-ay\frac{\partial}{\partial x}+\nu\nabla^2)\,\mathrm{d}t \tag{34}$$

Defining now the operators

$$A=\nu t\nabla^2,\quad B=-ayt\frac{\partial}{\partial x} \tag{35}$$

it is easy to see that

$$[A,[A,B]]=0 \text{ and } [B,[B,[A,B]]]=0 \tag{36}$$

and

$$[A,B]=-2\nu at^2\partial^2/\partial x\partial y \tag{37}$$

$$[B,[A,B]]=-2\nu a^2t^3\partial^2/\partial x^2 \tag{38}$$

hence, according to Eq. (B6),

$$G(x,t)=\exp(\nu t\nabla^2)\exp\left(-ayt\frac{\partial}{\partial x}\right)$$
$$\exp\left(\nu at^2\frac{\partial^2}{\partial x\partial y}\right)\exp\left(\frac{2}{3}\nu a^2t^3\frac{\partial^2}{\partial x^2}\right) \tag{39}$$

The order of the operators can be interchanged for computational convenience in accordance with the identity given in Eq. (B7). Thus,

$$G(x,t)=\exp(\nu t\nabla^2)\exp\left(\nu at^2\frac{\partial^2}{\partial y\partial x}\right)$$
$$\exp\left(\frac{1}{3}\nu a^2t^3\frac{\partial^2}{\partial x^2}\right)\exp\left(-ayt\frac{\partial}{\partial x}\right) \tag{40}$$

Solutions to Eqs. (32) and (33) can now be easily constructed:

$$v(x,t)=G(x,t)\,v(x,0) \tag{41}$$

$$w(x,t)=G(x,t)\,w(x,0) \tag{42}$$

Considerable physical insight can be gained by calculating the velocity correlation. Consider, for example,

$$\Gamma_{vv}(x,x',t)=\langle v(x,t)\,v(x',t)\rangle$$
$$=G(x,t)G(x',t)\langle v(x,0)\,v(x',0)\rangle \tag{43}$$

Since $G(x_i,t)$ and $G(x_i',t)$ commute,

$$\Gamma_{vv}(x,x',t)=\exp\left[\nu t(\nabla^2+\nabla'^2)\right]$$
$$\exp\left[\nu at^2\left(\frac{\partial^2}{\partial y\partial x}+\frac{\partial^2}{\partial y'\partial x'}\right)\right]\exp\left[\frac{1}{3}\nu a^2t^3\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x'^2}\right)\right]$$
$$\exp\left[-at\left(y\frac{\partial}{\partial x}+y'\frac{\partial}{\partial x'}\right)\right]\Gamma_{vv}(x_i,x_i',0) \tag{44}$$

In terms of the coordinates r_i and R_i given by Eqs. (21), the velocity correlation becomes

$$\Gamma_{vv}(r,R,t)=\exp\left[2\nu t\frac{\partial^2}{\partial r_i\partial r_i}\right]\exp\left[2\nu at^2\frac{\partial^2}{\partial r_1\partial r_2}\right]$$
$$\exp\left[\frac{2}{3}\nu a^2t^3\frac{\partial^2}{\partial r_1^2}\right]\exp\left[-atr_2\frac{\partial}{\partial r_1}\right]\exp\left[\frac{1}{2}\nu t\frac{\partial}{\partial R_i\partial R_i}\right]$$
$$\exp\left[\frac{1}{2}\nu at^2\frac{\partial^2}{\partial R_1\partial R_2}\right]\exp\left[\frac{1}{6}\nu a^2t^3\frac{\partial^2}{\partial R_1^2}\right]$$
$$\exp\left[-atR_2\frac{\partial}{\partial R_1}\right]\Gamma_{vv}(r,R,0) \tag{45}$$

The corresponding expression in Fourier space has the form

$$\Gamma_{vv}(k,K,t)=\exp\left[-2\nu t(k^2+ak_2k_1t+\frac{a^2}{3}k_1^2t^2)\right]$$
$$\exp\left[-\frac{1}{2}\nu t(K^2+aK_2K_1t+\frac{a^2}{3}K_1^2t^2)\right]$$
$$\times{}_{vv}(k_1,k_2+atk_1,k_3,k_1,k_2+atK_1,K_3;0) \tag{46}$$

For homogeneous turbulence the above result may be further simplified as follows:

$$\Gamma_{vv}(k,t)=\exp[-2\nu t(k^2+ak_2k_1t+\frac{a^2}{3}k_1^2t^2)]$$
$$\times\Gamma_{vv}(k_1,k_2+atk_1,k_3,0) \tag{47}$$

Except for a factor k^{-4} which arises from pressure, this is identical to the result derived by Deissler[10] and Hasen.[11]

It is immediately seen that for a uniform mean flow, that is for $a=0$ the velocity correlation and, similarly, the turbulent kinetic energy, which is a summation of terms like $\Gamma_{vv}(k_i,t)$, are attenuated exponentially with respect to time; furthermore, the attenuation is more violent for large wave numbers, that is, for small eddies. For sheared flow, though, it is possible that the turbulent energy increases with time for a while, if the product k_2k_1 is negative. However, the term $a^2/3\ k^2t^2$ will soon overpower the term ak_2k_1t and the energy will eventually start decreasing exponentially.

For a physically reasonable initial condition, Deissler[10] proposes a correlation that corresponds to isotropic turbulence.[8] Specifically,

$$\Gamma_{ij}(k,0)=(J_0/12\pi^2)(\delta_{ij}-k_ik_j) \tag{48}$$

For this particular case, our Eq. (47) becomes

$$\Gamma_{vv}(k,t)=\frac{J_0}{12\pi^2}(k_1^2+k_3^2)\exp[-2\nu t(k^2+ak_2k_1t+\frac{a^2}{3}k_1^2t^2)] \tag{49}$$

Similar results have been derived by Hasen.[11]

Comparing the method employed by Deissler[10] with ours, we observe that he solves equations for the second moment and then only in the Fourier space. In fact, his method is confined to linear mean flow profiles, because only then can he arrive at partial differential equations of the first order, that have closed form solutions. If a quadratic profile is assumed, then one would come up with second derivatives in phase-space. In the present method and at the same level of approximation, we work directly with equations for the random fluctuations. The solutions are then combined in order to form second moments. Moreover our analysis proceeds in the physical space and only for purposes of comparison with earlier theories we calculate the Fourier transform of the final result. Deissler's work is confined to homogeneous flows. The present analysis is not limited to such flows and again, only for the sake of comparison, our result is reduced to Deissler's. Finally, the present method can take into account the time dependence of the mean flow.

The solution of Eq. (31) is a little more complicated, since it involves a convolution integral that depends on $v(x,t)$:

$$u(x,t)=G(x,t)u(x,0)-a\int_0^t G(x,t-s)\,v(x,s)\,ds \tag{50}$$

This expression can be used to calculate the Reynolds stress:

$$\Gamma_{uv}(x,x',t)=\langle u(x,t)\,v(x',t)\rangle=G(x,t)\langle u(x,0)\,v(x,0)\rangle$$
$$-a\int_0^t G(x,t-s)\langle v(x,s)\,v(x',t)\rangle ds \tag{51}$$

The calculation of the velocity correlation in the x direction involves double convolution integrals and a numerical intergration would be necessary.

V. Time-Dependent Sheared Mean Flow

We consider here a mean flow with a time-dependent shear. Specifically, we assume that

$$U_1(x,y,z,t)=ayf(t),\quad U_2=0,\quad U_3=0 \tag{52}$$

With $E(t)=-ayf(t)\partial/\partial x+\nu\nabla^2$, we calculate the functions $\Delta_i(t,t_0)$ (cf. Appendix A):

$$\Delta_1(t,0)=-ayg_1(t)\frac{\partial}{\partial x}+\nu t\nabla^2 \tag{53}$$

$$\Delta_2(t,0) = \nu a g_2(t) \frac{\partial^2}{\partial x \partial y} \tag{54}$$

$$\Delta_3(t,0) = -2\nu a^2 g_3(t) \frac{\partial^2}{\partial x^2} \tag{55}$$

$$\Delta_n(t,0) = 0 \text{ for } n \geq 4 \tag{56}$$

where

$$g_1(t) = \int_0^t f(s)\mathrm{d}s \tag{57}$$

$$g_2(t) = \int_0^t [s_1 f(s_1) - \int_0^{s_1} \mathrm{d}s_2 f(s_2)]\mathrm{d}s_1 \tag{58}$$

$$g_3(t) = \frac{1}{6}\int_0^t \int_0^{s_1} \int_0^{s_2} \{[f(s_1) - f(s_2)]f(s_3) + [f(s_3) - f(s_2)]f(s_1)\}\mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \tag{59}$$

Our solution can be constructed in terms of the propagator

$$G(x,t) = \exp\left[\nu t \nabla^2 - a y g_1(t) \frac{\partial}{\partial x} + \nu a g_2(t) \frac{\partial^2}{\partial x \partial y} - 2\nu a^2 g_3(t) \frac{\partial^2}{\partial x^2}\right] \tag{60}$$

Once again we can factorize this expression according to the identify given by Eq. (B6):

$$G(x,t) = \exp(\nu t \nabla^2)\exp(-a y g_1(t) \frac{\partial}{\partial x}) \exp\left[\nu a (t g_1(t) + g_2(t)) \frac{\partial^2}{\partial x \partial y}\right]\exp\left[-\frac{1}{2}\nu a^2 g_2(t) g_1(t) \frac{\partial^2}{\partial x^2}\right] \exp\left[-\frac{2}{3}\nu a^2 t g_1^2(t) \frac{\partial^2}{\partial x^2}\right]\exp\left[-2\nu a^2 g_3(t) \frac{\partial^2}{\partial x^2}\right] \tag{61}$$

Interchanging the order of the exponential operators according to Eq. (B7), we finally arrive at the expression

$$G(x,t) = \exp(\nu t \nabla^2)\exp\left[\nu a (t g_1(t) + g_2(t)) \frac{\partial^2}{\partial x \partial y}\right] \exp\left\{\nu a^2 [\frac{1}{2} g_2(t) g_1(t) + \frac{1}{3} t g_1^2(t) - 2 g_3(t)] \frac{\partial^2}{\partial x^2}\right\}\exp(-a y g_1(t) \frac{\partial}{\partial x}) \tag{62}$$

Let us again calculate, for homogeneous trubulent flow, a component of the velocity correlation in the Fourier space:

$$\Gamma_{vv}(k,t) = \exp\left\{-2\nu t[k^2 + a g_1(t) k_1 k_2] - 2\nu a g_2(t) k_1 k_2 - 2\nu a^2 k_1^2 [\frac{1}{2} g_2(t) g_1(t) + \frac{1}{3} t g_1^2(t) - 2 g_3(t)]\right\} \times \Gamma_{vv}(k_1, k_2 + a g_1(t) k_1, k_3, 0) \tag{63}$$

We consider next the following special cases:

a) $f(t) = 1 + \alpha t$.

This corresponds to a linear shear profile that becomes steeper and steeper as time increases. In this case the functions g_i become

$$g_1(t) = t + \alpha t^2/2, \; g_2(t) = \alpha t^3/6, \tag{64a}$$

$$g_3(t) = -\alpha^2 t^5/240 \tag{64b}$$

and for homogeneous flow, Eq. (63) assumes the form

$$\Gamma_{vv}(k,t) = \exp\left\{-2\nu t[k^2 + a k_1 k_2 (t + \frac{2\alpha}{3} t^2)] - 2\nu a^2 k_1^2 (\frac{1}{3} t^3 + \frac{5\alpha}{12} t^4 + \frac{2}{15}\alpha^2 t^5)\right\} \Gamma_{vv}(k_1, k_2 + a g_1(t) k_1, k_3, 0) \tag{65}$$

Once again, it is observed that the term $a k_1 k_2 (t + (2\alpha/3)t^2)$ may permit a temporary increase with time. This will be quickly dominated by higher powers of time and the solution will eventually decay. It is interesting to note that for small mean flow shear, that is for small a, this effect is even more pronounced. The growth of this term is further reinforced due to the unsteady effects, that is the presence of the term $2\alpha t^2/3$. It may be possible that this growth may violate the validity of the present analysis, before the higher order terms take over. Such an instability effect could conceivably be the catalytic effect for the creation of large turbulent eddies, a phenomenon that we can not capture with our low turbulence Reynolds number approximation.

b) $f(t) = 1 + \alpha \sin(\omega_0 t)$.

This corresponds to a linear shear profile with a slope that oscillates with frequency ω_0 about a mean equal to 1. The functions g_i, after some algebra, are found to be

$$g_1(t) = t - \frac{\alpha}{\omega_0}\cos\omega_0 t + \frac{\alpha}{\omega_0} \tag{66}$$

$$g_2(t) = \frac{2\alpha}{\omega_0^2}\sin(\omega_0 t) - \frac{\alpha}{\omega_0} t\cos(\omega_0 t) - \frac{\alpha}{\omega_0} t \tag{67}$$

$$g_3(t) = -\frac{\alpha}{\omega^3} - \frac{3}{8}\frac{\alpha^2 t}{\omega^2} + \frac{1}{12}\frac{\alpha t^2}{\omega} + \frac{\alpha}{\omega^3}\cos\omega t + \frac{1}{2}\frac{\alpha^2}{\omega^2}\sin\omega t + \frac{1}{8}\frac{\alpha^2}{\omega^3}\sin 2\omega t - \frac{1}{24}\frac{\alpha^2}{\omega^2} t\cos 2\omega t - \frac{1}{3}\frac{\alpha^2}{\omega^2} t\cos t\omega + \frac{1}{2}\frac{\alpha}{\omega^2} t\sin\omega t - \frac{1}{12}\frac{\alpha}{\omega} t^2\cos\omega t \tag{68}$$

The exponential of Eq. (63) now becomes

$$\frac{1}{2} g_2 g_1 + \frac{1}{3} t g_1^2 - 2 g_3 = \frac{1}{3} t^3 + \frac{\alpha^2}{2\omega_0^2} + \frac{2\alpha}{\omega_0^3} - \frac{2\alpha}{\omega_0^3}\cos\omega_0 t - \frac{3}{4}\frac{\alpha^2}{\omega_0^3}\sin 2\omega_0 t + \frac{\alpha^2}{\omega_0^2} t\cos^2\omega_0 t - \frac{\alpha}{\omega_0} t^2\cos\omega_0 t \tag{69}$$

The dominant term remains the term $1/3t^3$ and the solution decays again with time.

Finally, we can consider a mean flow fluctuation independent of space

$$U_1(x,y,z,t) = ay + \alpha\sin(\omega_0 t), \; U_2 = 0, \; U_3 = 0 \tag{70}$$

This case can be treated relatively easy with the present method. On the other hand it is quite significant in practice, because it has been observed that the deterministic part of fluctuating turbulent boundary layers is almost uniform across the thickness of the boundary layer[21] at least for large values of the frequency.

The propagator G, by virtue of the Eq. (B6), becomes

$$G(x,t) = \exp\left[-\int_0^t \alpha \sin\omega_0 t \frac{\partial}{\partial x} dt\right] \exp\left[\int_0^t \left(ay\frac{\partial}{\partial x} + \nu\nabla^2\right)dt\right] \tag{71}$$

and the velocity correlation reads

$$\Gamma_{vv}(x,x',t) = G(x,t)G(x',t) \quad \Gamma_{vv}(x,x',0)$$

$$= \exp\left[-\frac{\alpha}{\omega_0}(1-\cos\omega_0 t)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial x'}\right)\right]\exp\left[t\left(ay\frac{\partial}{\partial x}+\nu\nabla^2\right)\right]$$

$$\exp\left[t\left(ay'\frac{\partial}{\partial x'}+\nu\nabla'^2\right)\right]\Gamma_{vv}(x,x',0) \tag{72}$$

Taking a Fourier transform of the above expression we arrive at an equation almost identical to Eq. (46)

$$\Gamma_{vv}(k,K,t) = \exp\left[-2\nu t\left(k^2 + ak_2k_1t + \frac{a^2}{3}k_1^2t^2\right)\right]$$

$$\exp\left[-\frac{1}{2}\nu t\left(K^2 + aK_2K_1t + \frac{a^2}{3}K_1^2t^2\right) + \frac{\alpha}{\omega_0}(1-\cos\omega_0 t)L_1\right]$$

$$\Gamma_{vv}\left(k_1,k_2,k_3,K_1,K_2+\frac{\alpha}{\omega_0}(1-\cos\omega_0 t)K_1,K_3,0\right) \tag{73}$$

The new term represents a frequency modulation proportional to the amplitude and inversely proportional to the frequency of the forced fluctuation. The effect disappears if homogeneity is assumed.

VI. Conclusions and Recommendations

This paper represents the initial results of an effort to understand and utilize some powerful mathematical tools in the theory of of turbulence. Several classical results have been reproduced by a more effective and straightforward method. Unlike previous methods, the present approach is not confined to steady mean flows or homogeneous turbulence.

In the spirit of the present paper, that is, within the low Reynolds number approximation, some natural and straightforward extensions are possible. For example, we intend to extend the present theory to a quadratic mean profile, a profile that resembles more realistically the flow in a turbulent boundary layer. Then we intend to consider the effect of a wall which would give us some information about the viscous sublayer and the inner layer of a turbulent boundary layer. It is felt that in this region the linear profile assumption and the small disturbance approximation are quite realistic but the pressure fluctuations play a vital role and cannot be ignored.

This investigation will hopefully guide the parallel effort of improving the closure assumptions that are necessary for calculations of engineering problems. It is later intended to improve the level of approximation by taking into account terms proportional to the square of fluctuating quantities. In this case though some numerical calculations will be necessary in order to yield results for comparison with experimental data.

Appendix A

Throughout the present paper we use methods of solution that have been developed in the last few decades and widely used in physics and quantum mechanics. They include exponential operators, time-ordering functions, etc. These are tools that have been proven to be powerful and effective in certain areas, yet we found that they are relatively unknown to the engineering community. In this Appendix, and the one that follows, we review some of the most basic ideas and present several useful formulas and identities. For more details the reader is referred to Wilcox[20] and Misguich and Balescu.[14,15]

Consider the differential equation

$$\frac{\partial u(x,t)}{\partial t} = E(x,t)u(x,t) + F(x,t) \tag{A1a}$$

$$u(x,t_0) = u_0(x) \tag{A1b}$$

where $E(x_i,t)$ is a differential operator containing space derivatives only. The solution can be written as follows:

$$u(x,t) = G(x,t,t_0)u_0(x) + \int_{t_0}^t G(x,t,t')F(x,t')dt' \tag{A2}$$

The propagator $G(x,t,t_0)$ is a space operator, defined as the solution of an initial value problem, viz.

$$\frac{\partial}{\partial t}G(x,t,t_0) = E(x,t)G(x,t,t_0), \qquad t \geq t_0 \tag{A3a}$$

$$G(x,t_0,t_0) = I \tag{A3b}$$

where I is the identity operator. For an unbounded domain, $G(x,t,t_0)$ is given explicitly by the formula

$$G(x,t,t_0) = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{t_0}^t dt_1\int_{t_0}^{t_1}dt_2\ldots\int_{t_0}^{t_{n-1}}dt_n$$

$$XE(x,t_1)E(x,t_2)\ldots E(x,t_n) = X\exp\left[\int_{t_0}^t E(x,s)ds\right] \tag{A4}$$

The symbol X in this expression denotes a time-ordering operator and is defined by

$$XE(x,t_1)E(x,t_2)\ldots E(x,t_n) = E(x,t_{k1})E(x,t_{k2})\ldots Ex,t_{kn}) \tag{A5}$$

where $t_{k1} \geq t_{k2} \geq \ldots \geq t_{kn}$.

In general, the propagator $G(x,t,t_0)$ can be found by an expansion of the form

$$G(x,t,t_0) = \exp\sum_{n=1}^{\infty}\Delta_n(x,t,t_0) \tag{A6}$$

The operators $\Delta_n(x_j,t,t_0)$ can be expressed in terms of the operator $E(x_i,t)$ as follows:

$$\Delta_1(x,t,t_0) = \int_{t_0}^t E_1 dt_1 \tag{A7}$$

$$\Delta_2(x,t,t_0) = \frac{1}{2}\int_{t_0}^t dt_1\int_{t_0}^{t} dt_2[E_1,E_2] \tag{A8}$$

$$\Delta_3(x,t,t_0) = \frac{1}{6}\int_{t_0}^t dt_1\int_{t_0}^{t_1}dt_2\int_{t_0}^{t_2}dt_3$$

$$\{[[E_1,E_2],E_3] + [[E_3,E_2],E_1]\} \tag{A9}$$

$$\Delta_4(x,t,t_0) = \frac{1}{12}\int_{t_0}^t dt_1\int_{t_0}^{t_1}dt_2\int_{t_0}^{t_2}dt_3\int_{t_0}^{t_3}dt_4$$

$$\{[[[E_3,E_2],E_4],E_1] + [[[E_3,E_4],E_2],E_1]$$

$$+ [[[E_1,E_2],E_3],E_4] + [[[E_4,E_1],E_3],E_2]\} \tag{A10}$$

where E_n denotes $E(x,t_n)$ and bracket symbols stand for the commutator of two operators. It can be seen that if E does not depend on time, the time-ordering operator reduces to the identity operator.

Appendix B

This Appendix includes a number of basic formulas and identities of the algebra of exponential operators. It is provided here with the intention of familiarizing the reader with some elementary operations used in the present paper.

As a simple example of an exponential operator, consider the operator $E=c\partial/\partial x$, with c a constant. The exponential operator in (A4) corresponds in this case to a summation of derivatives, viz.

$$G(x,t,t_0)=e^{c\partial/\partial x}=1+(t-t_0)\partial/\partial x+(1/2)(t-t_0)^2\partial^2/\partial x^2 \tag{B1}$$

Similarly, for the operator $E=C_i\partial/\partial x_i$, with C_j a constant vector, the exponential operator in (A4) corresponds to a spatial displacement. Specifically,

$$G(x,t,t_0)=\sum_{n=0}^{\infty}\frac{1}{n!}\left[(t-t_0)C_i\frac{\partial}{\partial x_i}\right]^n \tag{B2}$$

and

$$G(x,t,t_0)f(x,t)=f[x+(t-t_0)C,t] \tag{B3}$$

It should be noted, however, that if E is a more complex operator, the interpretation of the exponential operator in (A4) is not always obvious. For example, for a Laplacian operator $c\partial^2/\partial x_i\partial x_i$, with c a constant, the propagator in (A4) corresponds to an integral operator

$$\exp\left[c\frac{\partial^2}{\partial x_i\partial x_i}\right]f(x,t)=\frac{H(t)}{(4\pi c)^{3/2}}\int \mathrm{d}x'\exp\left[-\frac{(x-x')^2}{4c}\right]f(x,t) \tag{B4}$$

The exponential of a summation of two operators can be factorized only if the operators commute, namely

$$e^{A+B}=e^Ae^B \tag{B5a}$$

if

$$[A,B]=AB-BA=0 \tag{B5b}$$

If the operators A and B do not commute, then, in general,

$$\exp(A+B)=\exp A\ \exp B\ \exp(-\frac{1}{2}[A,B])\ \exp(\frac{1}{3}[B,[A,B]]+\frac{1}{6}[A,[A,B]])\ldots \tag{B6}$$

Moreover, if the operators A and B do not commute, the order of the product of exp A and exp B can be interchanged as follows:

$$\exp A\ \exp B=\exp B\ \exp A\ \exp([A,B]) \tag{B7a}$$

if

$$[A,[A,B]]=[B,[A,B]]=0 \tag{B7b}$$

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